Boğaziçi University

ECONFIN - MATHEMATICS WAIVER EXAM

Solutions

14/07/2016

You have 80 minutes! Good Luck!!

NAME:

1. (20 pts) Consider the following matrices.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{bmatrix}, \ D = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

Compute each of the the following. If it is not computable, explain why.

(a) $A\ B\ C$

Solution:

$$A\ B\ C = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 21 & 16 \\ 8 & 8 \end{bmatrix}$$

(b) $(A^T)\ (B^T)\ C$

Solution:

Not computable. The number of columns in B^T is not equal to the number of rows in C.

(c) Inverse of D

Solution:

$$D^{-1} = \frac{1}{det(D)} \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -3/2 \\ 0 & 1 \end{bmatrix}$$

(d) $(A + C)^T + B$

Solution:

$$(A + C)^T + B = \begin{bmatrix} 3 & 0 \\ 4 & 4 \\ 0 & 4 \end{bmatrix}^T + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 1 \\ 4 & 4 & 6 \end{bmatrix}$$

2. (10 pts) Find the determinant of the following matrix

$$D = \begin{bmatrix} 0 & 3 & 2 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

Solution:

$$det(D) = 0 \cdot det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 3 \cdot det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} + 2 \cdot det \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = (-3) \cdot 2 + 2 \cdot (-3) = -12$$

3. (10 pts) Solve the following system of equations for x, y, z, using Cramer's Rule.

$$\begin{aligned} x + 2y - z &= 7 \\ 2y - z &= 5 \\ x - y + 4z &= 3 \end{aligned}$$

Solution:

Denote the system as

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$

Then, $det A = 7$.

By Cramer's Rule,

$$x = \tfrac{1}{7}\ det \begin{bmatrix} 7 & 2 & -1 \\ 5 & 2 & -1 \\ 3 & -1 & 4 \end{bmatrix} = \tfrac{1}{7}14 = 2$$

$$y = \tfrac{1}{7}\ det \begin{bmatrix} 1 & 7 & -1 \\ 0 & 5 & -1 \\ 1 & 3 & 4 \end{bmatrix} = \tfrac{1}{7}21 = 3$$

$$z = \tfrac{1}{7}\ det \begin{bmatrix} 1 & 2 & 7 \\ 0 & 2 & 5 \\ 1 & -1 & 3 \end{bmatrix} = \tfrac{1}{7}7 = 1$$

4. (20 pts) Differentiate the following functions with respect to x.

(a) $f(x) = \frac{x^3+2}{1-x^2}$

Solution:

$f'(x) = \frac{3x^2(1-x^2)-(x^3+2)(-2x)}{(1-x^2)^2} = \frac{3x^2-x^4+4x}{(1-x^2)^2}$

(b) $f(x) = (5x^2-3)^2(\ln(x^3))$

Solution:

$$\begin{aligned} f'(x) &= \ln(x^2) \cdot [2(5x^2-3) \cdot 10x] + 3x^2 \frac{1}{x^3} \cdot (5x^2-3)^2 \\ &= (5x^2-3)[\ln(x^2) \cdot 20x + \frac{3}{x}(5x^2-3)] \end{aligned}$$

5. (10 pts) Find the total differential of the following function and evaluate it when $x = 2$ and $dx = 0.02$.

$$y = 3x^4 - x^3 + 2x^2 - 1$$

Solution:

Note that $dy = (12x^3 - 3x^2 + 4x)dx$. Then when $x = 2$ and $dx = 0.02$ we get $dy = (12 \cdot 8 - 3 \cdot 4 + 4 \cdot 2) \cdot 0.02 = 92 \cdot 0.02 = 1.84$

6. (14 pts) Solve the following integrals. Use integration by parts if needed.

(a) $\int (\frac{1}{x} + 2e^{-x} + x^4)dx$

Solution:

$\int (\frac{1}{x} + 2e^{-x} + x^4)dx = \ln(x) - 2e^{-x} + \frac{x^5}{5} + C$

(b) $\int_1^e 2x \ln(x)dx$

Solution:

$\int_1^e 2x \ln(x)dx = \int_1^e udv$

where $u = \ln(x)$ and $dv = 2xdx$.

So, $du = (1/x)dx$ and $v = \int 2xdx = x^2$

Then, by integration by parts we get

$\int_1^e 2x \ln(x)dx = \int_1^e udv = uv - \int_1^e vdu = [x^2 \ln(x)]_1^e - \int_1^e x^2 \frac{1}{x}dx = [e^2 \ln(e) - 1^2 \ln(1)] - \int_1^e xdx$

Then,

$\int_1^e 2x \ln(x)dx = e^2 - \int_1^e xdx = e^2 - [\frac{x^2}{2}]_1^e = e^2 - [e^2/2 - (1/2)] = \frac{e^2+1}{2}$

7. (16 pts) Suppose that a person has the utility function, $u(x, y) = x^{1/2}y^{1/2}$, where x is the amount of good x and y is the amount of good y. The consumer's budget constraint is $2x + y = 100$. Using the Lagrange method find the optimal amounts of consumption for both goods that maximize the utility. Check second order test to verify the extremum points are indeed maximum.

Solution: